Algorithms for k-meet-semidistributive lattices

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October 12, 2015

Abstract

In this paper we consider k-meet-semidistributive lattices and we are interested in the computation of the set-colored poset associated to an implicational base. The parameter k is of interest since for any finite lattice $\mathcal{L}$ there exists an integer k for which $\mathcal{L}$ is k-meet-semidistributive. When $k = 1$ they are known as meet-semidistributive lattices.

We first give a polynomial time algorithm to compute an implicational base of a k-meet-semidistributive lattice from its associated colored poset. In other words, for a fixed k, finding a minimal implicational base of a k-meet-semidistributive lattice L from a context (FCA literature) of L can be done not just in output-polynomial time (which is open in the general case) but in polynomial time in the size of the input. This result generalizes that in [26]. Second, we derive an algorithm to compute a set-colored poset from an implicational base which is based on the enumeration of minimal transversals of an hypergraph and turns out to be in polynomial time for k-meet-semidistributive lattices [20, 13]. Finally, we show that checking whether a given implicational base describes a k-meet-semidistributive lattice can be done in polynomial time.

Keywords: k-meet-semidistributive lattice, colored poset, implicational base.

1 Introduction

Finite lattices representations (or duality) have been studied in many types of frameworks: posets for distributive lattices [9], binary relations (or contexts) and set-colored posets for general lattices [29, 30, 6, 19, 24], implicational bases (or Horn expressions), closure systems [12]. Computing a representation from a given one has been investigated in several areas of mathematics

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and computer science such as Logic, Lattice theory, Database, Graphs and Hypergraphs.

Research around implicational bases has lead to several notions for "good" bases. We distinguish the Duquenne-Guiges implicational base which is minimum [15], the canonical direct base [8, 38] which corresponds to minimal generators. There is a refinement of the canonical direct base that is generally smaller and, when ordered, remains direct [3]. Other notions of implicational bases have been recently considered in [1, 2, 35].

Computing an implicational base from a binary relation (or a context in FCA terminology) is a central problem for applications like data mining [21, 33], FCA [14], artificial intelligence [16, 17], game theory [22, 34], databases [4, 11, 10], integer linear programming [11, 10]. In [2, 35], the enumeration of minimal transversals of an hypergraph is used to compute an implicational base without considering the k-meet-semidistributivity. The reader is referred to [37] for a nice survey on this topic and, particularly, the use of minimal transversals of an hypergraph. Concerning the complexity, Babin and Kuznetsov [5] have shown that deciding whether an implication belongs to a minimum implicational base is a coNP-complete problem. The existence of an output-polynomial time algorithm for the computation of a minimum implicational base remains an open question [5]. Still, for hypergraphs, the problem of enumerating all minimal transversals in hypergraphs is a well known special case, which has been shown to be quasi-polynomial in the size of the input and output [18]. Only few cases with polynomial time algorithms have been considered in the literature: distributive, meet-distributive and meet-semidistributive lattices [25, 26], modular lattices [39].

Computing the binary relation corresponding to an implicational base has been less considered in the literature. In the eighties, Beeri et al. [7], and Mannila and Raihä [28] considered the Armstrong relation corresponding to a set of functional dependencies on a set of attributes. Kavvadias et al. [27] have shown that the problem of computing the maximal models corresponding to maximal meet-irreducible cannot be solved in output-polynomial time for arbitrary implication bases or Horn expressions, unless P=NP. Still, for the general case, the computation of all meet-irreducible elements (not necessarily maximal) from an implicational base remains an open problem. The number of meet-irreducible elements can be exponential in the number of maximal meet-irreducible.

In this paper we consider k-meet-semidistributive lattices [20, 13] and we are interested in the computation of the set-colored poset associated to an implicational base. The parameter k is of interest since for any finite lattice $\mathcal{L}$ there exists an integer k where $\mathcal{L}$ is k-meet-semidistributive. For $k = 1$ they are known as meet-semidistributive.

We first give a polynomial time algorithm to compute an implicational base of a k-meet-semidistributive lattice from its associated colored poset and therefore its binary representation. This result generalizes a former result from Janssen and Nourine [26]. In other words, for a fixed k, finding a minimal implicational base of a k-meet-semidistributive lattice L from a set-colored poset (or a context in FCA literature) of L can be done not just in output-polynomial time (which is open in the general case) but in polynomial time in the size of the input. Then we give an algorithm based on the enumeration of minimal transversals of a hypergraph to find all meet-irreducible elements from an implicational base which turns out to be in polynomial time for k-meet-semidistributive lattices. Then, this algorithm allows us to construct the set-colored poset associated to the implicational base. Finally, we provide a polynomial time algorithm to decide whether a given implicational base describes a k-meet-semidistributive.

2 Preliminaries

For classic vocabulary around lattices, the reader may refer to [12, 19]. Nevertheless, we provide here some notations that we use in this paper.

Given a lattice $\mathcal{L} = (E, \wedge, \vee, \leq)$, the set of its join-irreducible elements is $J(\mathcal{L})$. Similarly, $M(\mathcal{L})$ denotes the set of its meet-irreducible elements. Moreover, if x is an element of E, the set $J(x)$ is the set of all join-irreducible elements that are smaller than or equal to x. The set $\downarrow x$ is the set of all elements (not necessarily join-irreducible) that are smaller than or equal to x (this is the *ideal* of x). When X is a set of elements of E, we refer to $J(X)$ for $\bigcup_{x \in X} J(x)$ and $\downarrow X$ for $\bigcup_{x \in X} \downarrow x$. The notations $M(x)$, $M(X)$, $\uparrow x$ and $\uparrow X$ refer to meet-irreducible elements and *filters*. For a join-irreducible element j in $J(\mathcal{L})$, its unique predecessor is denoted by j_* (one can notice that j_* is not a join-irreducible element in general). Dually m^* denotes the unique successor of a meet-irreducible element m.

Colored posets have been introduced by Habib and Nourine [24, 32, 23, 31] to capture structural properties of lattices. We will use the arrow relations introduced by Wille [40]. Given a lattice $\mathcal{L}$, a join-irreducible element j and a meet-irreducible element m, we say that j has *color* m if j is a minimal element in $\mathcal{L}$ restricted to $E \setminus \downarrow m$. We shall note $j \swarrow m$ when it happens. If m is a maximal element in $\mathcal{L}$ restricted to $E \setminus \uparrow j$, we note $j \nearrow m$. Whenever $j \swarrow m$ and $j \nearrow m$, we say that m is a *principal color* of j and note $j \swarrow\!\!\!\!\nearrow m$. The set of colors of an element j is denoted by $\gamma(j)$. For any set X of join-irreducible elements, $\gamma(X)$ is defined as $\bigcup_{j \in X} \gamma(j)$.

The *colored poset* $P(\mathcal{L}) = (J(\mathcal{L}), \leq, \gamma, M(\mathcal{L}))$ associated to a lattice $\mathcal{L}$ is the poset restricted to its join-irreducible elements together with the sets of colors $\gamma(j)$ for each join-irreducible element j. A subset C of $M(\mathcal{L})$ is said to be an *ideal color set* if there exists an ideal I of P such that $\gamma(I) = C$. The set of all ideal color sets of $P(\mathcal{L})$, denoted by $\mathcal{C}(\mathcal{L})$ is a coclosure system isomorphic to $\mathcal{L}$ [24]. Consider the application $g : \mathcal{C}(\mathcal{L}) \to 2^{J(\mathcal{L})}$ which associates to each C in $\mathcal{C}(\mathcal{L})$ the maximal ideal $g(C) = \{j \in J(\mathcal{L}) \mid \gamma(J(j)) \subseteq$

$C\}$, and denote by $\mathcal{IM}(\mathcal{L}) = \{g(C) \mid C \in \mathcal{C}(\mathcal{L})\}$. Then $\mathcal{L}$ is isomorphic to $\mathcal{C}(\mathcal{L})$ (or $\mathcal{IM}(\mathcal{L})$) when ordered under set containment.

An *implicational base* Σ on a set J is a set of pairs (A, B) in $2^J \times 2^J$, denoted, by $A \to B$. The Σ-closure of a subset X of J is the minimal set X^Σ containing X such that for any rule $A \to B$ in Σ, if A is included in X^Σ, then B is also included in X^Σ. The set of all Σ-closed sets is denoted by $\mathcal{C}(\Sigma)$, which is a closure system. Clearly there are several implication bases with the property that $\mathcal{C}(\Sigma)$ is isomorphic $\mathcal{IM}(\mathcal{L})$ when ordered under set inclusion. A set $\Sigma_\mathcal{L}$ is said to be an *implicational base of* $\mathcal{L}$ if $\mathcal{C}(\Sigma_\mathcal{L}) = \mathcal{IM}(\mathcal{L})$.

Using the isomorphism between $\mathcal{L}$ and the closure systems $\mathcal{C}(\Sigma_\mathcal{L})$ and $\mathcal{IM}(\mathcal{L})$, we will identify an element x in E with the set $J(x)$.

Definition 1. [20] A lattice $\mathcal{L}$ is said to be *k-meet-semidistributive* if each join-irreducible element has at most k principal colors. Furthermore, when k equals 1, we simply say that $\mathcal{L}$ is *meet-semidistributive*.

Note also that any lattice $\mathcal{L}$ is k-meet-semidistributive for some k smaller than or equal to the number of meet-irreducible elements of $\mathcal{L}$, meaning that $k \leq |M(\mathcal{L})|$. In the rest of this paper we consider the following problems:

BASE
Input: the colored poset $P(\mathcal{L})$ for some k-meet-semidistributive lattice $\mathcal{L}$.
Output: an implicational base $\Sigma_\mathcal{L}$ of $\mathcal{L}$.

COLOR POSET
Input: an implicational base $\Sigma_\mathcal{L}$ for some k-meet-semidistributive lattice $\mathcal{L}$.
Output: the colored poset $P(\mathcal{L})$.

k-MEET-SEMIDISTRIBUTIVE RECOGNITION
Input: an implicational base Σ for some lattice $\mathcal{L}$, and k an integer.
Output: YES, if $\mathcal{L}$ is k-meet-semidistributive, NO otherwise.

3 Computing an implicational base from a colored poset representing a k-meet-semidistributive lattice

We first derive an algorithm that computes an implicational base from a colored poset (or a context) representing a lattice. It generalizes the result given by Janssen and Nourine in [26] for 1-meet-semidistributive lattices.

Given a lattice $\mathcal{L}$, a join-irreducible element j and a set of join-irreducible elements A, we define the set $P_{j,A}$ as follows:

$$P_{j,A} = \{x \in J(\mathcal{L}) \mid x < j \text{ or there exists } j' \text{ in } A \text{ such that } x \leq j'\}.$$

Lemma 2. *Let $\mathcal{L}$ be a lattice, j a join-irreducible element and A a set of join-irreducible elements. If the principal colors of j are contained in $\gamma(A)$, then $\bigvee P_{j,A} \geq j$.*

Proof. For a contradiction, suppose first that $\bigvee P_{j,A} < j$. In this case, every element j' of A satisfies $j' \leq j_*$. If m is any color of j then $j' \leq j_* \leq m$, and so m cannot be a color of j'.

Now suppose that $\bigvee P_{j,A}$ and j are incomparable. Let m be a maximal element in $\uparrow \bigvee P_{j,A} \setminus \uparrow j$. Then m is meet-irreducible and $j \nearrow m$. From $m \geq \bigvee P_{j,A} \geq j_*$ follows $j \swarrow m$, hence m is a principal color of j. Because m is larger than every element of A (in view of $m \geq \bigvee P_{j,A}$), it follows that m is not in $\gamma(A)$.

□

In our approach of k-meet-semidistributive lattices, it will be useful to define the following implicational systems. For any lattice $\mathcal{L}$ and any integer k, let

$$\Sigma_1(\mathcal{L}) = \{j \to j' : j \text{ covers } j' \text{ in } J(\mathcal{L})\},$$

$$\Sigma_2^k(\mathcal{L}) = \{P_{j,A} \to j : j \in J(\mathcal{L}), A \subseteq J(\mathcal{L}), |A| \leq k, \text{ and} \bigvee P_{j,A} \geq j\} \text{ and}$$

$$\Sigma_c^k(\mathcal{L}) = \Sigma_1(\mathcal{L}) \cup \Sigma_2^k(\mathcal{L}).$$

Theorem 3. *For any integer k and any k-meet-semidistributive lattice $\mathcal{L}$, the set of implications $\Sigma_c^k(\mathcal{L})$ is an implicational base for $\mathcal{C}(\mathcal{L})$.*

Proof. Let X be an element in $\mathcal{C}(\mathcal{L})$. There exists x in E such that $X = J(x)$. The set X is closed under $\Sigma_1(\mathcal{L})$ since it only translates the partial order reduced to join-irreducible elements. Moreover, X is also closed under $\Sigma_2^k(\mathcal{L})$ since these implications have been defined such that they comply with lattice $\mathcal{L}$.

Let X be a set of join-irreducible elements closed under $\Sigma_c^k(\mathcal{L})$. For a contradiction, suppose it is not in $\mathcal{C}(\mathcal{L})$ and denote by X' the smallest element of $\mathcal{C}(\mathcal{L})$ containing X ($X' = J(\bigvee X)$). Let j be a minimal element of $X' \setminus X$. Let m be a principal color of j. Since j is in X', X' cannot be below m. Then, X cannot be included in $J(m)$ otherwise its closure X' would be below m. Let j_m be a minimal element in $X \setminus J(m)$. Then m is in $\gamma(j_m)$. Therefore, every principal color of j is a color of some element of X. Call A the set $\{j_m : m \text{ is a principal color of } j\}$. Then $P_{j,A}$ is included in X and has size smaller than k since $\mathcal{L}$ is k-meet-semidistributive. By Lemma 2, $\bigvee P_{j,A} \geq j$. This contradicts the fact that X is closed under $\Sigma_2^k(\mathcal{L})$. □

Janssen and Nourine [26] have proved this same result in the case of meet-semidistributive lattices, by restricting to sets A of size 1. Their result can be reformulated as follows.

Theorem 4. *[26, Theorem 1] Let $\mathcal{L}$ be a meet-semidistributive lattice. Then the set of implications $\Sigma_c^1(\mathcal{L})$ is an implicational base for $\mathcal{C}(\mathcal{L})$.*

In order to compute this base, we only check whether $P_{j,A} \geq j$. This can easily be put in terms of colored posets. We just check that $\gamma(j) \subseteq \gamma(P_{j,A})$. When restricted to k-meet-semidistributive lattices, all theses operations can be carried out in polynomial time with respect to the number of join-irreducible elements. Moreover, we can use the algorithm in [36] to obtain a minimum one.

Corollary 5. *Given a colored poset of a k-meet-semidistributive lattice. Computing a minimum implicational base can be done in polynomial time in the sum of k and the size of the colored poset.*

Remark 6. It is worth noticing that Theorem 3 can be applied to general lattices. Indeed, given a colored poset of a lattice $\mathcal{L}$, we can compute in polynomial time the smallest k for which $\mathcal{L}$ is k-meet-semidistributive. A natural question is whether known algorithms for lattices (e.g. Nextclosure) can be parametrized by k.

4 Computing the colored poset for k-meet-semidistributive lattices

Let Σ be an implicational base on $J(\mathcal{L})$ for some lattice $\mathcal{L}$. We propose an algorithm based on the enumeration of minimal transversals of hypergraphs to compute the set of meet-irreducible elements of $\mathcal{C}(\Sigma)$.

This algorithm is not polynomial for general lattices with k unbounded, but when restricted to k-meet-semidistributive lattices with a bounded k, it is polynomial.

For this section, we will often use the identification of an element x and the set $J(x)$ already explained in section 2. To this end, a meet-irreducible element m of $\mathcal{L}$ can be seen as the set M in $\mathcal{C}(\Sigma)$, where M is made of all join-irreducible elements that are below m.

Let $P(\Sigma) = (J(\mathcal{L}), \leq, \gamma, N)$ be the colored poset associated to Σ. The order $(J(\mathcal{L}), \leq)$ can be computed as follows: $j \leq j'$ if and only if Σ satisfies the rule $j' \rightarrow j$. To compute the coloring γ, we first compute the set of meet-irreducible elements. Each meet-irreducible element can be seen as a set of join-irreducible elements. Let $\mathcal{M}(\Sigma)$ denote these meet-irreducible elements. For any join-irreducible element j, we define the set $\mathcal{M}_j(\Sigma)$ as the maximal sets in $\mathcal{M}(\Sigma)$ which include $J(j) \setminus \{j\}$ and do not contain j.

Lemma 7. *Let $\mathcal{L}$ be a lattice and (j, m) an element of $J(\mathcal{L}) \times M(\mathcal{L})$. Then $j \swarrow\!\!\!\nearrow m$ if and only if $J(m)$ is in $\mathcal{M}_j(\Sigma)$.*

Given a subset A of $J(\mathcal{L})$, the procedure **FindMeet**(j, A) computes a meet-irreducible element M in $\mathcal{M}_j(\Sigma)$ containing A.

Algorithm 1: FindMeet(j, A)

Input: Σ an implicational base of a lattice $\mathcal{L}$
j, join-irreducible element of $\mathcal{L}$
A, subset of $J(\mathcal{L})$

Output: M, element of $\mathcal{M}(\Sigma)$ such that $M \cap J(j_*) = J(j_*) \setminus \{j\}$ and $A \subseteq M$
$\emptyset$, if no such element exists

```
begin
    if j ∈ A^Σ then
        Return(∅);
    Let X = A ∪ J(j_*);
    for j' ∈ J(L) \ (X ∪ ↑j) do
        if j ∉ (X ∪ {j'})^Σ then
            X = X ∪ {j'};
    Return(X);
end
```

Lemma 8. ***FindMeet***(j, A) *returns an element of* $\mathcal{M}_j(\Sigma)$ *that contains* A *if such an element exists and* $\emptyset$ *otherwise.*

Proof. Notice first that there exists C in $\mathcal{L}$ such that $A \subseteq C$ and $j \notin C$ if and only if j is not in A^Σ and then **FindMeet**(j, A) returns $\emptyset$ if and only if $C(j, A) := \{C \in \mathcal{L} \mid A \subseteq C \text{ and } j \notin C\} = \emptyset$. Assume now that $C(j, A)$ is not empty, then clearly **FindMeet**(j, A) returns a maximal element M of $C(j, A)$ that contains $J(j_*)$. So let us show that $\max\{C \in C(j, A) : J(j_*) \subseteq C\} = \mathcal{M}_j(\Sigma)$ where max denotes the maximal sets with respect to set containment. Clearly $\mathcal{M}_j(\Sigma) \subseteq \max\{C \in C(j, A) : J(j_*) \subseteq C\}$. Let M be an element of $\max\{C \in C(j, A) : J(j_*) \subseteq C\}$, then M is maximal in $\mathcal{L} \setminus \uparrow j$. Thus M is in $\mathcal{M}(\Sigma)$ since otherwise $\mathcal{L}$ would not be a lattice. Thus, $\max\{C \in C(j, A) : J(j_*) \subseteq C\} = \max\{C \in C(j, A) \mid J(j_*) \subseteq C\} \cap \mathcal{M}(\Sigma) = \mathcal{M}_j(\Sigma)$. □

We use the idea of minimal transversal of an hypergraph to enumerate all elements M in $\mathcal{M}_j(\Sigma)$. Assume that we have already enumerated $M_1, ..., M_i$ elements of $\mathcal{M}_j(\Sigma)$ and denote by $\mathcal{H}_j^i$ the hypergraph with vertices $J(\mathcal{L}) \setminus \uparrow j$ and edges the sets $\{J(\mathcal{L}) \setminus M_1, \dots, J(\mathcal{L}) \setminus M_i\} \subset \mathcal{M}_j(\Sigma)$. The following theorem shows that there exists a new M in $\mathcal{M}_j(\Sigma)$ and a minimal transversal A of $\mathcal{H}_j^i$ such that $A \subseteq M$.

Lemma 9. *Let* $\{M_1, ..., M_i\}$ *be a subset of* $\mathcal{M}_j(\Sigma)$*. Then for all* M *in* $\mathcal{M}_j(\Sigma) \setminus \{M_1, ..., M_i\}$ *there exists a minimal transversal* A *of* $\mathcal{H}_j^i$ *such that* $A \subseteq M$*. Moreover* ***FindMeet***(j, A) *returns an element of* $\mathcal{M}_j(\Sigma) \setminus \{M_1, ..., M_i\}$*.*

Proof. Consider the hypergraph $\mathcal{H}_j^i = (J(\mathcal{L})\setminus \uparrow j, \{J(\mathcal{L})\setminus M_1, ..., J(\mathcal{L})\setminus M_i\})$ and let M be an element of $\mathcal{M}_j(\Sigma) \setminus \{M_1, ..., M_i\}$. Since for every k in $\{1, 2, \dots, i\}$ M is not a subset of M_k, there must exist x in M such that x is not in M_k and thus x is in $J(\mathcal{L}) \setminus M_k$. Then M is a transversal of $\mathcal{H}_j^i$ and there is a minimal transversal A of $\mathcal{H}_j^i$ with $A \subseteq M$.

Moreover, by Lemma 8, **FindMeet**(j, A) computes an element M of $\mathcal{M}_j(\Sigma)$ containing A. Since A is a transversal of $\mathcal{H}_j^i$, for any k in $\{1, 2, \dots, i\}$ A is not a subset of M_k. Thus for any such k $M \neq M_k$, *i.e.* M is in $\mathcal{M}_j(\Sigma) \setminus \{M_1, ..., M_i\}$. □

Algorithm 2: All-Meet$(\Sigma, J(\mathcal{L}))$

Input: an implicational base Σ on $J(\mathcal{L})$ of a lattice $\mathcal{L}$
Output: the set of all meet-irreducible $\mathcal{M}(\Sigma)$ of $\mathcal{C}(\Sigma)$
begin
  $\mathcal{M}(\Sigma) = \emptyset$;
1 **for** $j \in J(\mathcal{L})$ **do**
    $\mathcal{H}_j = (J(\mathcal{L})\setminus \uparrow j, \emptyset)$;
    $Continue = true$;
2   **while** $Continue = true$ **do**
      $Continue = false$;
      $Temp = \emptyset$;
3   **for** *each minimal transversal* A *of* $\mathcal{H}_j$ **do**
        $M =$**FindMeet**(j, A);
        if $M \neq \emptyset$ **then**
          $Temp = Temp \cup \{J(\mathcal{L}) \setminus M\}$;
          Add M to $\mathcal{M}(\Sigma)$;
          $Continue = true$;
      Add the hyperedges in $Temp$ to $\mathcal{H}_j$;
  Return$(\mathcal{M}(\Sigma))$;
end

Theorem 10. *Let Σ be an implicational base on J representing some lattice $\mathcal{L}$. Then Algorithm* ***All-Meet***$(\Sigma, J(\mathcal{L}))$ *returns* $\mathcal{M}(\Sigma)$.

Proof. It suffices to show that the content of the **for**-loop 1 computes $\mathcal{M}_j(\Sigma)$ for each j in $J(\mathcal{L})$ since for all M in $\mathcal{M}(\Sigma)$ there exists j such that M is in $\mathcal{M}_j(\Sigma)$. Indeed let M be an element of $\mathcal{M}(\Sigma)$ and let M^* be the unique successor of M in $\mathcal{L}$. Then there exists j in $M^* \setminus M$, furthermore M is maximal in $J(\mathcal{L})\setminus \uparrow j$, *i.e.* M is in $\mathcal{M}_j(\Sigma)$. So let j be a join-irreducible element of $\mathcal{L}$ and let us show that $\mathcal{M}_j(\Sigma)$ is computed inside the **for**-loop 1.

By Lemma 8, each produced element $M :=$**FindMeet**(j, A) belongs to $\mathcal{M}_j(\Sigma)$. Assume now that the set $\{M_1, ..., M_k\} \subseteq \mathcal{M}_j(\Sigma)$ has been computed (*i.e.* $\{J(\mathcal{L}) \setminus M_1, ..., J(\mathcal{L}) \setminus M_k\}$ is the current set of hyperedges of $\mathcal{H}_j$) and assume that $\{M_1, ..., M_k\} \neq \mathcal{M}_j(\Sigma)$. Then by Lemma 9, there exists a minimal transversal A of $\mathcal{H}_j$ such that **FindMeet**(j, A) returns an element of $\mathcal{M}_j(\Sigma) \setminus \{M_1, ..., M_k\}$. Thus the **for**-loop 3 will produce an element of $\mathcal{M}_j(\Sigma) \setminus \{M_1, ..., M_k\}$. This shows that new elements of $\mathcal{M}_j(\Sigma)$ are produced until each element has been computed. □

Lemma 11. *Let Σ be an implicational base on $J(\mathcal{L})$ of a k-meet-semidistributive lattice $\mathcal{L}$. Then Algorithm **All-Meet**$(\Sigma, J(\mathcal{L}))$ computes $\mathcal{M}(\Sigma)$ in $O(|J(\mathcal{L})| \times k \times |J(\mathcal{L})|^k)$.*

Proof. The **for**-loop 1 makes $|J(\mathcal{L})|$ iterations and for each step the **for**-loop 3 will be called at most $|\mathcal{M}_j(\Sigma)|$ times which is bounded by k. The **for**-loop 3 itself makes as many steps as the number of minimal transversal of the current hypergraph $\mathcal{H}_j$. Since $\mathcal{H}_j$ is of size at most k, the number of its minimal transversal is bounded by $|J(\mathcal{L})|^k$. Then the total complexity of **All-Meet** is $O(|J(\mathcal{L})| \times k \times |J(\mathcal{L})|^k)$. □

Given the set $\mathcal{M}(\Sigma) = \{M_1, ..., M_m\}$, we construct the colored poset $P(\Sigma) = (J(\mathcal{L}), \gamma, N)$ where $N = \{1, 2, ..., m\}$ and i is a color of j if and only if j is minimal in $J(\mathcal{L}) \setminus M_i$.

Algorithm 3: Colored-Poset$(\Sigma, J(\mathcal{L}))$

Input: an implicational base Σ on $J(\mathcal{L})$ of a lattice $\mathcal{L}$
Output: the colored poset $P(\Sigma) = (J(\mathcal{L}), \gamma, N)$ representing the lattice $\mathcal{L}$

begin
  $\mathcal{M}(\Sigma) =$ **All-Meet**$(\Sigma, J(\mathcal{L}))$;
  Assume $\mathcal{M}(\Sigma) = \{M_1, ..., M_m\}$;
  for $j \in J(\mathcal{L})$ **do**
    $\gamma(j) = \emptyset$;
1 **for** $M_i \in \mathcal{M}(\Sigma)$ **do**
    for $j \in J(\mathcal{L})$ *such that j is minimal in $J(\mathcal{L}) \setminus M$* **do**
      $\gamma(j) = \gamma(j) \cup \{i\}$;
end

Theorem 12. *Let Σ be an implicational base on $J(\mathcal{L})$ of a lattice $\mathcal{L}$. Then closed sets of Σ are exactly ideal colors sets of the colored poset $P(\Sigma)$, i.e. $\mathcal{C}(\Sigma) = \mathcal{IM}(\mathcal{L})$.*

Proof. Recall that every element $J(\mathcal{L})$ corresponds to a unique join-irreducible in $\mathcal{C}(\Sigma)$ when ordered under inclusion.

Assume that C is an ideal color set of $P(\Sigma)$ and $I = \{j \in J(\mathcal{L}) : \gamma(\downarrow j) \subseteq C\}$ the corresponding ideal in $P(\Sigma)$. Let us show that I is closed under Σ, *i.e.* I is in $\mathcal{C}(\Sigma)$. Suppose that $A \to j$ is an implication of Σ with $A \subseteq I$ and j not in I. Then there exists a color c in $\gamma(j)$ but not in $\gamma(I)$. Thus for any j' in I, c is not in $\gamma(j')$ which implies that j' is in M_c and then $I \subseteq M_c$. This contradicts the fact that $A \to j$ is an implication of Σ since $I^\Sigma \subseteq M_c$ and j is not in M_c.

Conversely, Let $I \subseteq J(\mathcal{L})$ be a closed set of Σ. Then there exists meet-irreducible $M_{i_1}, ..., M_{i_l}$ in $\mathcal{M}(\Sigma)$ such that $I = M_{i_1} \cap ... \cap M_{i_l}$. Clearly I is an ideal of $P(\Sigma)$. Suppose that I is not in $\mathcal{IM}(\mathcal{L})$. Then there exists j in $J(\mathcal{L}) \setminus I$ such that $J(j_*)$ is a subset of I and $\gamma(j)$ is a subset of $\gamma(I)$.

Let us show that for every M_c in $\{M_{i_1}, ..., M_{i_l}\}$ $I \subseteq M_c$ implies that j is in M_c and then conclude that j is in $I = M_{i_1} \cap ... \cap M_{i_l}$.

Suppose that there exists M_c in $\{M_{i_1}, ..., M_{i_l}\}$ such that j is not in M_c. Then c is in $\gamma(\downarrow j)$, but since $J(j_*) \subseteq M_c$, c must be in $\gamma(j)$. Thus there exists j' in I such that c is in $\gamma(j')$ which contradicts the fact that j' is in M_c. $\square$

Algorithm **Colored-Poset** is interesting in the sense that when we restrict the input to k-meet-semidistributive lattices (for some fixed integer k), it has a polynomial time complexity.

Theorem 13. *Let Σ be an implicational base for some k-meet-semidistributive lattice $\mathcal{L}$. Then Algorithm **Colored-Poset**$(\Sigma, J(\mathcal{L}))$ computes the colored poset $P(\Sigma)$ in $O(|J(\mathcal{L})| \times k \times |J(\mathcal{L})|^k)$.*

Proof. The complexity of **Colored-Poset** is the complexity of **All-Meet** plus $O(|J(\mathcal{L})| \times |\mathcal{M}(\Sigma)|)$ which is bounded by $O(|J(\mathcal{L})|^2 \times k)$. Since by Lemma 11 the complexity of **All-Meet** is bounded by $O(|J(\mathcal{L})|^2 \times k)$ we obtain the announced result.

$\square$

5 Recognition of k-meet-semidistributive lattices

We are now able to recognize k-meet-semidistributive lattices using polynomial time complexity.

Theorem 14. *For any fixed integer k, one can efficiently decide if an implicational base describes a k-meet-semidistributive lattice.*

Proof. We simply use Algorithm **Colored-Poset** with a restriction in our call to the subroutine **All-Meet**. This is indeed the time-consuming routine in the main algorithm. Since we are only interested in finding k-meet-semidistributive lattices, we know that the total number of meet-irreducible elements cannot be more than $k \cdot |J(\mathcal{L})|$. Therefore, as soon as Algorithm **All-Meet** finds $k \cdot |J(\mathcal{L})| + 1$ meet-irreducible elements, we may break the routine and answer that the given base does not describe a k-meet-semidistributive lattice.

If it gives less meet-irreducible elements, then the whole algorithm turns in polynomial time and we consider the colored poset that is built.

We can easily compute the principal colors $\{j \updownarrow m \mid m \in M(\mathcal{L})\}$ for each join-irreducible element $j \in J(\mathcal{L})$. If every join-irreducible element has less than k principal colors, the base describes a k-meet-semidistributive lattice. If not, it does not. $\square$

Theorem 15. *One can decide if an implicational base Σ on a set J describes a meet-semidistributive lattice in time $\mathcal{O}(|J|^3)$.*

Proof. Following the scheme described above, the computation of the set-colored poset should not take more than $\mathcal{O}(|J|^2)$. After this step we have at most $|J|$ different sets forming $\mathcal{M}(\Sigma)$. For each join-irreducible element j, we have to check that there is a single maximal element in $\gamma(j)$. This is made easily by taking the union of all its colors and checking if it is among its colors. This certifies that j has exactly one principal color. Computing the union of at most $|J|$ sets of size at most $|J|$ takes at most $\mathcal{O}(|J|^2)$ steps and checking if the computed set appears in $\gamma(j)$ takes the same time. In the end the whole algorithm has a time complexity of $\mathcal{O}(|J|^3)$.

$\square$

6 Open question

Clearly our strategy captures structural properties of lattices according to the number of principal colors for a join-irreducible. This means that if the number of principal colors of elements in a colored poset $P = (J(\mathcal{L}), \leq, \gamma, M)$ is bounded by a constant, then all considered problems are in polynomial time. As shown by Wild [39], modular lattices are k-meet-semidistributive lattices with k is unbounded, but computing an implicational base can be done in polynomial time from a colored poset. So, the natural question is: *for which classes of lattices with unbounded k these problems are polynomial.* Recall that distributive, meet-distributive and meet-semidistributive have $k = 1$.

Geyer [20] proved that a finite k-meet-semidistributive lattice can be characterized by a finite list a forbidden sublattices if and only if $k = 1$. We are convinced that the parameter k is of interest in lattice theory and algorithmic aspects to address these open problems. This paper answers the question when k is constant.

Acknowledgement

The authors wish to thank the anonymous referees for helpful comments and pointing out enlightening references.

The first author is supported by the Région Auvergne through the project M2F. The third author is supported by CNRS (Mastodons PETASKY project, 2012-2015).

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